A Full Analytical Solution to the Direct and Inverse Kinematics of the Pentaxis Robot Manipulator

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2014 Published by $\mathcal{DIFU}_{100}ci@$ http://nautilus.uaz.edu.mx/difu100cia

Abstract

In this paper, the kinematics of a 5-DOF robot manipulator was obtained. The direct kinematics was presented using the Denavit-Hartenberg convention and method. The inverse kinematics was obtained analytically and all possible solutions were found for each joint. The kinematic solution was verified experimentally by programming a kinematic solver for the MoveIt! ROS® platform.

Keywords: Kinematics, Robot Manipulator, 5-DOF, ROS

1. Introduction

The PENTAXIS robot was developed in Centro de Investigación y Desarrollo de Tecnología Digital (CITEDI-IPN) in 1994 for industrial and educational purposes and since then has been acquired by other research Institutes in the region. Computation of direct and inverse kinematics plays an important role in trajectory generation, path or motion planning, position, and motion control of fully-actuated robots [1, 2].

The *direct-kinematics* consists in finding the base-coordinates of the end effector of the robot given the joint variables. The *inverse-kinematics* problem is opposite to the direct-kinematics problem, that is, it consists in determining the joint variables of a robot given the orientation and posture for the end effector in base coordinates [3].

There exist many methods to find the direct kinematics. For a robot with few joints (up to three), the direct kinematics can be analytically obtained. For a robot with higher degrees-of-freedom, the Denavit-Hartenberg method [4] becomes suitable. On the other hand, the computation of the inverse kinematics is more complicated since several solutions (posture) can be obtained. There exist several softwares that find or solve the inverse-kinematics problem as the default kinematic solver in *Moveit!* from ROS® [5] however, a solution cannot be found for certain Cartesian trajectories.

The contribution of this work is towards a construction of a software platform for the PENTAXIS under ROS® where it will help to program and record industrial tasks as path planning contouring, and peak-and-place opera-

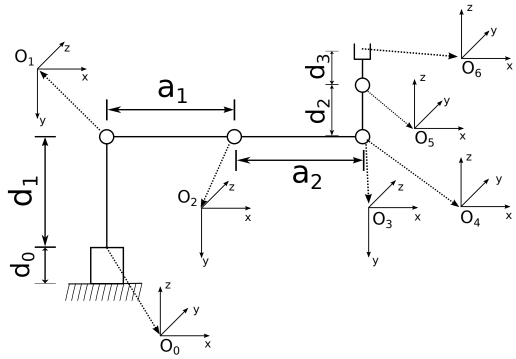

Figure 1. Coordinate frames for the PENTAXIS robot.

Figure 2. The PENTAXIS 5-DOF manipulator.

tions.

2. Anatomy of the Pentaxis Robot

The PENTAXIS is a 5-DOF robot manipulator with its joints of type revolute. When the robot is at its vertical pose, it will reach a height of 106 ± 0.05 cms if measured from its base to the tip of its end-effector which is a gripper. Every joint, including the end-effector, is driven by a stepper motor. The coordinate frames for the manipulator is shown in Fig. 1.

3. Forward Kinematics

The Denavit-Hartenberg (D-H) convention was used to derive the direct kinematics. The direct kinematics is a set of equations which is used to calculate the position and orientation of the end-effector from a given joint values by applying the D-H method to the coordinate frames depicted in Fig. 1. The D-H parameters are shown in Table 1.

The transformation matrix between two consecutive frames, denoted as $\boldsymbol{T}$, is composed of the multiplication of the following homogeneous transformations matrices, denoted as $\boldsymbol{A}_i$:

Table 1. The D-H parameters of the PENTAXIS robot.

Frame$_i$	θ_i	$(d_i \pm 0.05)$ **cms**	$(a_i \pm 0.05)$ **cms**	α_i **deg**
0	-	15.2	0	0
1	θ_1	25.9	0	-90
2	θ_2	0	25.2	0
3	θ_3	0	25.0	0
4	θ_4	0	0	90
5	θ_5	3.5	0	0
6	-	12.1	0	0

$$\boldsymbol{A}_i = \mathbf{Rot}(z,\theta_i)\cdot\mathbf{Trans}(0,0,d_i)\cdot\mathbf{Trans}(a_i,0,0)\cdot\mathbf{Rot}(x,\alpha_i)$$
$$= \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where $\mathbf{Rot}(z,\theta_i)$ denotes the rotation of θ_i around the z axis, $\mathbf{Trans}(0,0,d_i)$ is the translation of d_i length, $\mathbf{Trans}(a_i,0,0)$ is the translation of a_i length, and $\mathbf{Rot}(x,\alpha_i)$ denotes the rotation of α_i around the x axis. For the PENTAXIS robot, the transformation matrix from the base (frame 0) to the final effector (frame 6) is specifically expressed as

$$\boldsymbol{T}_6^0 = \prod_{i=0}^{6}\boldsymbol{A}_i = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{n} = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}. \tag{2}$$

Here, $\boldsymbol{p} = [p_x \ p_y \ p_z]^T \in \mathbb{R}^3$ is the position vector and $\boldsymbol{n}$ is the 3×3 orientation matrix of the end-effector. The following set of equations are regarded as the direct kinematics for the PENTAXIS robot:

$$n_x = c_{234}c_1c_5 - s_1s_5 \tag{3}$$
$$n_y = c_{234}s_1c_5 + c_1s_5 \tag{4}$$
$$n_z = -s_{234}c_5 \tag{5}$$
$$o_x = -c_{234}c_1s_5 - s_1c_5 \tag{6}$$
$$o_y = -c_{234}s_1s_5 + c_1c_5 \tag{7}$$
$$o_z = s_{234}s_5 \tag{8}$$
$$a_x = s_{234}c_1 \tag{9}$$
$$a_y = s_{234}s_1 \tag{10}$$
$$a_z = c_{234} \tag{11}$$
$$p_x = c_1[a_2c_2 + a_3c_{23} + (d_5+d_6)s_{234}] \tag{12}$$
$$p_y = s_1[a_2c_2 + a_3c_{23} + (d_5+d_6)s_{234}] \tag{13}$$
$$p_z = d_0 + d_1 - a_2s_2 - a_3s_{23} + (d_5+d_6)c_{234} \tag{14}$$

where $c_i = \cos(\theta_i)$, $s_i = \sin(\theta_i)$, $c_{ij} = \cos(\theta_i+\theta_j)$, $s_{ij} = \sin(\theta_i+\theta_j)$, $c_{ijk} = \cos(\theta_i+\theta_j+\theta_k)$, and $s_{ijk} = \sin(\theta_i+\theta_j+\theta_k)$.

4. Inverse Kinematics

The analytic inverse kinematics consists of finding θ_1 to θ_5 algebraically from the set equations (3)–(14).

4.1. Solutions for θ_1

4.1.1. First solution

From equations (12) and (13) the following solution is obtained:

$$\theta_1 = \text{atan2}\left(\frac{p_y}{p_x}\right). \tag{15}$$

4.1.2. Second solution

A second solution for θ_1 can be obtained from (10) and (9), that is,

$$\theta_1 = \text{atan2}\left(\frac{a_y}{a_x}\right). \tag{16}$$

4.1.3. Third solution

Another solution may be derived, from equations (3) and (7) we first solve them for $s_1 s_5$, equate them, then substitute c_{234} from (11) and finally solve for $c_1 c_5$ we have:

$$c_1 c_5 = \frac{o_y - a_z n_x}{1 - a_z^2}. \tag{17}$$

Similar to (17), it is possible to obtain the following relation by now taking into account (4) and (6):

$$s_1 c_5 = \frac{-o_x - a_z n_y}{1 - a_z^2}. \tag{18}$$

Therefore, dividing (18) by (17) and solving for θ_1 we have a third solution:

$$\theta_1 = \text{atan2}\left(\frac{-o_x - a_z n_y}{o_y - a_z n_x}\right). \tag{19}$$

4.1.4. Fourth solution

A fourth solution may be obtained by applying a similar method for this last solution. Starting from (7) and (3), solving them for $c_1 c_5$, equate them, substitute c_{234} from (11) and finally solve for $s_1 s_5$ we have:

$$s_1 s_5 = \frac{n_x - a_z o_y}{a_z^2 - 1}. \tag{20}$$

We derive a similar equation as (20), by now taking into account (6) and (4) we have:

$$c_1 s_5 = \frac{-n_y - a_z o_x}{a_z^2 - 1}. \tag{21}$$

Dividing (20) by (21) and solving for θ_1 we have the fourth solution:

$$\theta_1 = \text{atan2}\left(\frac{n_x - a_z o_y}{-n_y - a_z o_x}\right). \tag{22}$$

4.1.5. Fifth solution

From equations (3) and (6) we solve for c_{234} and then equate them, thus:

$$s_1 = \frac{-s_5 n_x - c_5 o_x}{2}, \tag{23}$$

similarly from equations (4) and (7) we have:

$$c_1 = \frac{s_5 n_y + c_5 o_y}{2}. \tag{24}$$

Therefore from the previous two equations the fifth solution is obtained:

$$\theta_1 = \text{atan2}\left(\frac{-s_5 n_x - c_5 o_x}{s_5 n_y + c_5 o_y}\right). \tag{25}$$

In this last solution there is a dependency on θ_5, however there are solutions for θ_5 in which there is no dependence on any other unknown variable, as it is shown further in this document.

4.1.6. Sixth solution

When the arm of the robot is at the upright position the above solutions will have a “zero by zero division”. To avoid such singularity we set θ_5 to zero, therefore another solution is derived. From Eqs. (6) and (7) we have the following:

$$s_1 = -o_x, \quad c_1 = o_y,$$

therefore solution six is:

$$\theta_1 = \text{atan2}\left(\frac{-o_x}{o_y}\right). \tag{26}$$

4.2. Solutions for θ_5

4.2.1. First solution

From equations (8) and (5) we easily find the first solution for θ_5:

$$\theta_5 = \text{atan2}\left(-\frac{o_z}{n_z}\right). \tag{27}$$

4.2.2. Second solution

Equations (3) and (7) are solved for c_1c_5, equating them, substituting c_{234} from (11) and solving for s_1s_5 we have:

$$s_1s_5 = \frac{o_ya_z - n_x}{1 - a_z^2}. \tag{28}$$

Analogously to the last equation we derive a second one, by now taking into account (6) and (4) we have:

$$s_1c_5 = \frac{-o_x - a_zn_y}{1 - a_z^2}. \tag{29}$$

Dividing (28) by (29) and solving for θ_2 we have the second solution:

$$\theta_5 = \text{atan2}\left(\frac{o_ya_z - n_x}{-o_x - a_zn_y}\right). \tag{30}$$

4.2.3. Third solution

The derivation of the third solution for θ_3 is very similar from this last solution. From (7) we solve for s_1s_5, substitute that result into (3) and finally solve for c_1c_5, we have:

$$c_1c_5 = \frac{n_xa_z + o_y}{a_z^2 - 1}. \tag{31}$$

From (6) we solve for s_1c_5, substituting that result into (4) and finally solving for c_1s_5, we have:

$$c_1s_5 = \frac{-n_y - a_zo_x}{a_z^2 - 1}. \tag{32}$$

Dividing (32) by (31) and solving for θ_5 we finally have the third solution:

$$\theta_5 = \text{atan2}\left(\frac{-n_y - a_zo_x}{o_y + a_zn_x}\right). \tag{33}$$

4.2.4. Fourth solution

From equations (3) and (4) we solve for c_{234} and then equate them, thus:

$$s_5 = \left(\frac{c_1n_y - s_1n_x}{2}\right), \tag{34}$$

similarly from equations (6) and (7) we have:

$$c_5 = \left(\frac{c_1o_y - s_1o_x}{2}\right). \tag{35}$$

Therefore solution four is derived:

$$\theta_5 = \left(\frac{c_1n_y - s_1n_x}{c_1o_y - s_1o_x}\right). \tag{36}$$

4.3. Solutions for θ_3

From Eqs. (12)–(14) the following terms may be derived:

$$a_2c_2 + a_3c_{23} = \Gamma, \quad a_2s_2 + a_3s_{23} = \Omega, \tag{37}$$

where Γ is any of the following terms derived from the combination of (12) and (13) with (5), (8), (9), (10):

$$\begin{aligned}\Gamma &= \frac{p_x}{c_1} + (d_5 + d_6)\frac{n_z}{c_5} = \frac{p_x}{c_1} - (d_5 + d_6)\frac{o_z}{s_5}\\ &= \frac{p_x}{c_1} - (d_5 + d_6)\frac{a_x}{c_1} = \frac{p_x}{c_1} - (d_5 + d_6)\frac{a_y}{s_1}\\ &= \frac{p_y}{s_1} + (d_5 + d_6)\frac{n_z}{c_5} = \frac{p_y}{s_1} - (d_5 + d_6)\frac{o_z}{s_5}\\ &= \frac{p_y}{s_1} - (d_5 + d_6)\frac{a_x}{c_1} = \frac{p_y}{s_1} - (d_5 + d_6)\frac{a_y}{s_1},\end{aligned}$$

and Ω is any of the following terms derived from the combination of (14) with (3), (4), (6), (7), (11):

$$\begin{aligned}\Omega &= \epsilon + (d_5 + d_6)\frac{n_x + s_1s_5}{c_1c_5} = \epsilon + (d_5 + d_6)\frac{n_y - c_1s_5}{s_1c_5},\\ &= \epsilon + (d_5 + d_6)\frac{-o_x - s_1c_5}{c_1s_5} = \epsilon + (d_5 + d_6)\frac{-o_y + c_1c_5}{s_1s_5}\\ &= \epsilon + (d_5 + d_6)a_z,\end{aligned}$$

here $\epsilon = d_0 + d_1 - p_z$.
The square of the sum of Eqs. (37) will give us:

$$a_2^2 + 2a_2a_3(c_2c_{23} + s_2s_{23}) + a_3^2 = \Gamma^2 + \Omega^2. \tag{38}$$

Because $c_2c_{23} + s_2s_{23} = c_3$, we can write:

$$c_3 = \left(\frac{\Gamma^2 + \Omega^2 - a_2^2 - a_3^2}{2a_2a_3}\right), \quad s_3 = \pm\sqrt{1 - c_3{}^2}.$$

In the above equation, we interpret the positive square root for when the robot elbow (joint 3) is up and the negative for when the elbow is down. Thus we have:

$$\theta_3 = atan2\left(\frac{\pm\sqrt{1 - c_3{}^2}}{\left(\frac{\Gamma^2+\Omega^2-a_2^2-a_3^2}{2a_2a_3}\right)}\right). \tag{39}$$

4.4. Solutions for θ_2

From the preceding Subsection, Eqs. (37) may be expressed as follows:

$$(a_2{+}a_3c_3)c_2{-}(a_3s_3)s_2 = \Gamma, \quad (a_3s_3)c_2{+}(a_2{+}a_3c_3)s_2 = \Omega. \tag{40}$$

From above equations, we may find a solution for s_2 and c_3, and the combination of those solutions yield:

$$\theta_2 = \text{atan2}\left(\frac{(a_2 + a_3c_3)\Omega - (a_3s_3)\Gamma}{(a_2 + a_3c_3)\Gamma + (a_3s_3)\Omega}\right). \tag{41}$$

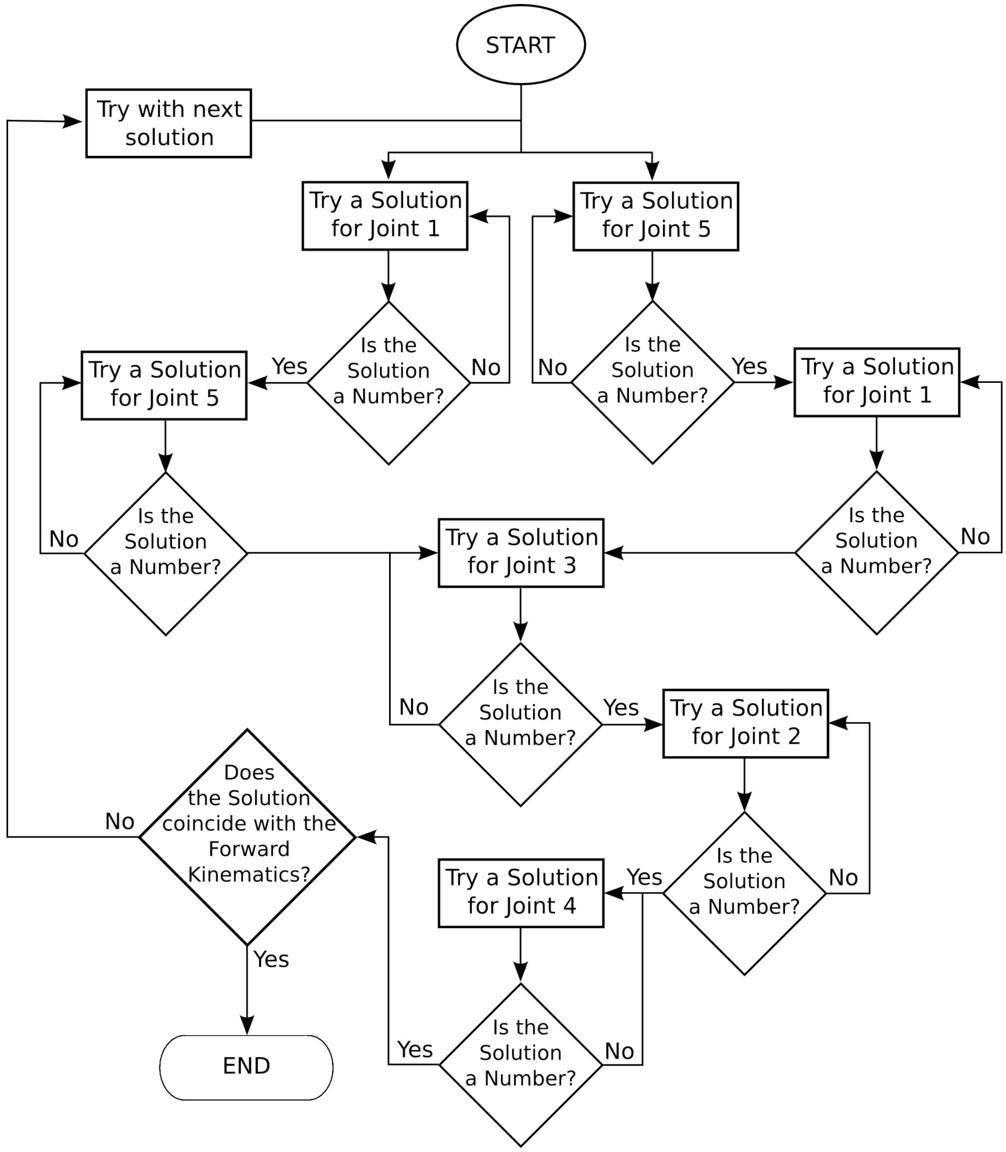

Figure 3. Flowchart of the Inverse Kinematic algorithm.

4.5. Solutions for θ_4

The solution of θ_4 can be straightforwardly found from the set of Eqs.

$$s_{234} = s_{23}c_4 + c_{23}s_4, \quad c_{234} = c_{23}c_4 - s_{23}s_4, \qquad (42)$$

where variables θ_2, θ_3 are known. By means of the many algebraic methods for solving linear equations, we find an equation for s_4 and another for c_4, dividing them yields:

$$\theta_4 = \operatorname{atan2}\left(\frac{c_{23}s_{234} - s_{23}c_{234}}{s_{23}s_{234} + c_{23}c_{234}}\right). \qquad (43)$$

Here, c_{234} and s_{234} is given by:

$$\begin{aligned} c_{234} &= \frac{n_x + s_1 s_5}{c_1 c_5} = \frac{n_y - c_1 s_5}{s_1 c_5} = \frac{-o_x - s_1 c_5}{c_1 s_5} = a_z \\ &= \frac{-o_y + c_1 c_5}{s_1 s_5} = \frac{[p_z - (d_0 + d_1) + a_2 s_2 + a_3 s_{23}]}{d_5 + d_6}, \\ s_{234} &= -\frac{n_z}{c_5} = \frac{o_z}{s_5} = \frac{1}{d_5 + d_6}\left(\frac{p_x}{c_1} - a_2 c_2 - a_3 c_{23}\right) \\ &= \frac{a_x}{c_1} = \frac{a_y}{s_1} = \frac{1}{d_5 + d_6}\left(\frac{p_x}{s_1} - a_2 c_2 - a_3 c_{23}\right). \end{aligned}$$

5. Implementation and experimental results

In order to test our results we created a MoveIt! Kinematics Plugin [6] for the ROS platform, then four different Cartesian paths was given for the robot's end effector to follow. The flowchart that represents the implemented algorithm for the inverse kinematics is shown in figure 3. We have let out the forward kinematics flowchart algorithm since its implementation is straightforward. A video has been recorded showing the robot following four paths: a rectangle, a circle, an ellipse, and a sine function. These paths were not possible to be calculated by the *MoveIt! Motion Planner* while using the default kinematic solver (the *KDL kinematics* numeric solver). The video is available in the following URL: `https://www.youtube.com/watch?v=Gl_mZR0w330`.

6. Conclusions

This document presents the forward and inverse kinematics of a 5-DOF robot manipulator. The forward kinematics was derived using the Denavit-Hartenberg notation, where a set of twelve equations were obtained. For the inverse kinematics many solutions were given for each joint, for which in some cases there will be redundancy and for other cases there is only a unique solution. The implementation for the direct kinematics is straightforward. However there may be many kinds of implementations for the inverse kinematics. In this paper, we used a backtrack kind algorithm, then from the set of solutions that were obtained, a comparison with the forward kinematics would indicate the correct one. This implementation of an analytic kinematic solver is novel in regards that it deals with redundancy and singularities, and also it is fast enough to come up with the solution so that the end user is able to interact with the ROS®'s Rviz visualizer in real time.

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